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Using vector divisions in solving the linear complementarity problem

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ABSTRACT

The linear complementarity problem $LCP(M, q)$ is to find a vector z in IR^n satisfying $z^T(Mz+q) = 0, Mz+q \geqslant 0, z \geqslant 0$, where $M = (m_{ij}) \in \mathrm{IR}^{n\times n}$ and $q \in \mathrm{IR}^n$ are given. In this paper, we use the fact that solving $LCP(M, q)$ is equivalent to solving the nonlinear equation $F(x) = 0$ where F is a function from IR^n into itself defined by $F(x) = (M+I)x+(M-I)|x|+q$. We build a sequence of smooth functions $\tilde{F}(p, x)$ which is uniformly convergent to the function $F(x)$. We show that, an approximation of the solution of the $LCP(M, q)$ (when it exists) is obtained by solving $\tilde{F}(p, x) = 0$ for a parameter p large enough. Then we give a globally convergent hybrid algorithm which is based on vector divisions and the secant method for solving $LCP(M, q)$. We close our paper with some numerical simulations to illustrate our theoretical results, and to show that this method can solve efficiently large-scale linear complementarity problems.

1. Introduction

The complementarity problem noted (CP) is a classical problem of the optimization theory of finding $(z, w) \in \mathrm{IR}^n \times \mathrm{IR}^n$ such that:

$$\begin{cases} \langle z, w\rangle = 0 \\ w - f(z) = 0 \\ z, \quad w \geqslant 0 \end{cases} \tag{1}$$

where f, a continuous operator from IR^n into itself, is given.

The constraint $\langle z, w\rangle = 0$ is called the complementarity condition since for any i, $1 \leqslant i \leqslant n,\ z_i = 0$ if $w_i > 0$ and vice versa. It may be the case when $z_i = w_i = 0$.

If f is a nonlinear continuous operator from IR^n into itself, the problem is called a *N*on *L*inear *C*omplementarity *P*roblem associated with the function f and noted $(NLCP)$. The problem is called a *L*inear *C*omplementarity *P*roblem (LCP) associated with the matrix M and the vector q if the function f is in the form:

$$f(z) = q + Mz,$$

where q is an element of IR^n and M is an $n \times n$ real matrix. The linear complementarity problem plays an important role in several fields such as operations research, game theory [1] and economic applications [2,3].

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It is known (see [4,5]) that all the problems of linear programming (*LP*), convex quadratic programming (*CQP*), and the problems of Nash equilibrium of a bi-matrix game can be written as a linear complementarity problems. It is difficult to solve (*LCP*) for any given function f. The existence and the characterization of solutions is not easy in the general case. A fundamental question is, under what conditions on the matrix M and the vector q this problem admits one and only one solution, and, if this is the case, how can we express this solution as a function of the matrix and vector mentioned above. This question has not been completely solved yet. However, many results already exist, for instance Lemke [5] give sufficient conditions on the matrix M and the vector q under which the number of solutions of $LCP(M, q)$ is finite. Samelson [6], Ingeton [7], Murty [8,9], Watson [10], Kelly [11] and Cottle [12] have by contrast shown that the matrix M is a *P-matrix* if and only if the linear complementarity problem associated with a matrix M and a vector q has a unique solution for all $q \in \mathrm{IR}^n$ (a matrix M is called a *P-matrix* if all principal minors are strictly positive (see [13]), recall that any symmetric and positive definite matrix is a *P-matrix*).

Many works are devoted to the construction of algorithms for solving linear and nonlinear complementarity problems in various situations (see [14] for a survey). One of the most popular methods is the interior point method (see [15–21] and references therein). Interior point algorithms are globally efficient and have good iteration complexity but they have, in general, the problem of finding a strictly feasible starting point.

In this work we propose a globally convergent hybrid algorithm for solving the Linear complementarity problem $LCP(M, q)$. We assume that the problem has a unique solution and we use the fact that solving $LCP(M, q)$ is equivalent to solving the nonlinear equation $F(x) = 0$ where F is a function from IR^n into itself defined by $F(x) = (M+I)x+(M-I)|x|+q$. We build a sequence of functions $\tilde{F}(p, x) \in C^\infty$ which uniformly converges to the function $F(x)$; and we show that finding the zero of the function F is equivalent to finding the zero of the sequence of the functions $\tilde{F}$.

The paper is organized as follows. In Section 2 we briefly give some definitions and notations to be used throughout the paper. In Section 3 we write LCP in the equivalent form of solving a nonlinear equation. In Section 4 we construct a sequence a functions giving an approximation of the nonlinear equation and introduce the algorithm. In Section 5 we give some numerical examples and we give conclusions in Section 6.

2. Preliminaries

This section is devoted to some notations and preliminary definitions.

Let IR^n be n-dimensional real Euclidean space and $\mathrm{IR}^{n\times n}$ be the set of all real $n \times n$ matrices.

We will use I to denote the identity matrix.

$x^T y$ or $\langle x, y\rangle$ is the inner product of the vectors $x, y \in \mathrm{IR}^n$; $\|x\|$ is the Euclidean norm.

For $x \in \mathrm{IR}^n$ and k a nonnegative integer, $x^{(k)}$ refers to the vector obtained after k iterations.

For $1 \leqslant i \leqslant n$, x_i refers to the ith element of x, and $x_i^{(k)}$ refers to the ith element of the vector obtained after k iterations.

$\mathrm{IR}^n_+ := \{x \in \mathrm{IR}^n : x_i \geqslant 0, i = 1 \dots n\}$ is the nonnegative orthant and its interior is $\mathrm{IR}^n_{++} := \{x \in \mathrm{IR}^n : x_i > 0, i = 1 \dots n\}$.

Let $x, y \in \mathrm{IR}^n$, the expression $x \leqslant y$ (respectively $x < y$) meaning that $x_i \leqslant y_i$ (respectively $x_i < y_i$) for each $i = 1 \dots n$.

The transpose of a vector x is denoted by x^T (with super script T).

For $x \in \mathrm{IR}^n$ we define $|x| = (|x_1|, \dots, |x_n|)^T \in \mathrm{IR}^n$, and we denote by $e^x = (e^{x_1}, \dots, e^{x_n})^T \in \mathrm{IR}^n$. For $x \in \mathrm{IR}^n_{++}$ we denote by $\ln(x) = (\ln(x_1), \dots, \ln(x_n))^T$.

Recall that the spectrum $\sigma(A)$ of the matrix A is the set of its eigenvalues and its spectral radius ρ is given by: $\rho(A) := \sup\{|\lambda| \text{ such that } \lambda \in \sigma(A)\}$.

3. Equivalent reformulation of LCP

It is well known (see [22]) that the linear complementarity problem $LCP(M, q)$ is completely equivalent to solving the nonlinear equation

$$F(x) = 0$$

where F is a function from IR^n into itself defined by

$$F(x) := (M + I)x + (M - I)|x| + q.$$

More precisely (see [22]), if x^* is a zero of the function F, and

$$\begin{cases} z^* := |x^*| + x^* \\ w^* := |x^*| - x^* \end{cases} \tag{2}$$

then (z^*, w^*) is a solution of $LCP(M, q)$.

Conversely, if (z^*, w^*) is a solution of $LCP(M, q)$, then

$$x^* := \frac{z^* - w^*}{2}$$

is a zero of the function F.

The equation $F(x) = 0$ can be solved by the fixed point algorithm (see [23]), this algorithm is defined by:

$$\begin{cases} x^{(0)} \in \mathrm{IR}^n \text{ arbitrary}, \\ x^{(k+1)} = (I+M)^{-1}(I-M)|x^{(k)}| - (I+M)^{-1}q. \end{cases} \tag{3}$$

For the case that M is symmetric and positive definite, it was shown in [24] (see also Section 9.2 in [22]) that, for

$$D := (I+M)^{-1}(I-M)$$

we have

$$\|D\|_2 = \sqrt{\rho(D^T D)} = \sqrt{\rho(D^2)} = \rho(D) < 1,$$

hence, by the contraction-mapping theorem [25], the algorithm (3) converges and

$$x^* = \lim_{k\to+\infty} x^{(k)}$$

is the unique solution of the $F(x) = 0$.

Therefore,

$$w^* := |x^*| - x^*$$

and

$$z^* := |x^*| + x^*$$

defines the unique solution of the $LCP(M, q)$.

The convergence of the algorithm (3) is only linear. To improve the speed of convergence, it is obvious that we need to replace the above function F by a smooth one. In this paper, we exhibit a sequence of smooth functions converging to F and we use the vector divisions with the secant method (see [26]) to find an approximation of the solution of $F(x) = 0$. The obtained algorithm is globally convergent (see [26]).

4. The main result

Let $F : \mathrm{IR}^n \to \mathrm{IR}^n$ defined by

$$F(x) := (M+I)x + (M-I)|x| + q.$$

Consider the sequence of functions $\tilde{F} : \mathrm{IN}^* \times \mathrm{IR}^n \to \mathrm{IR}^n$ defined by

$$\tilde{F}(p, x) := (M+I)x + \frac{1}{p}(M-I)\ln(e^0 + e^{px} + e^{-px}) + q.$$

Then we have

Proposition 1. *$\tilde{F}(p, x)$ converges uniformly to $F(x)$ when $p \to +\infty$.*

Proof. We have:

$$\begin{aligned} \frac{1}{p}\ln(e^0 + e^{px} + e^{-px}) - |x| &= \frac{1}{p}[\ln(e^0 + e^{px} + e^{-px}) + \ln(e^{-p|x|})] \\ &= \frac{1}{p}\ln[e^{-p|x|} * (1 + e^{px} + e^{-px})] \\ &= \frac{1}{p}\ln(e^{-p|x|} + e^{p(x-|x|)} + e^{-p(x+|x|)}). \end{aligned}$$

Then

$$0 \leqslant \frac{1}{p}\ln(e^0 + e^{px} + e^{-px}) - |x| \leqslant \frac{1}{p}\ln(\mathbf{3})$$

where $\mathbf{3}$ is the vector $(3, \ldots, 3)^T$; so,

$$\frac{1}{p}\ln(e^0 + e^{px} + e^{-px})$$

convergences uniformly to $|x|$ when $p \to +\infty$.

Moreover, the operator $(M - I)$ is linear, so we have

$$\frac{1}{p}(M-I)\ln(e^0 + e^{px} + e^{-px}) \text{ converges uniformly to } (M-I)|x| \quad \text{as } p \to +\infty$$

and from the expression of the sequence of the functions $\tilde{F}$ and the function F we have

$$\tilde{F}(p, x) \text{ converges uniformly to } F(x) \quad \text{when } p \to +\infty.$$

Theorem 2. *Let $x^*(p)$ be a solution of the equation $\tilde{F}(p,x) = 0$, then $x^*(p)$ is an approximate solution of $F(x) = 0$ for p large enough.*

Proof. From the proposition above we have $\forall \epsilon > 0$, $\exists p^* > 0$ such that for all $p > p^*$:

$$\begin{aligned}\|F(x^*(p))\| &= \|F(x^*(p)) - \tilde{F}(p, x^*(p))\| \\ &\leqslant \epsilon.\end{aligned}$$

So, $x^*(p)$ is an approximate solution of

$$F(x) = 0.$$

Remark 1. The uniqueness of the root of the function F results from the uniqueness of the solution of the linear complementarity problem $LCP(M,q)$; in fact, if x_1^* and x_2^* are two distinct roots of the function F, then

$$z_1^* := |x_1^*| + x_1^*.$$
$$z_2^* := |x_2^*| + x_2^*.$$

Since $z_1^* = z_2^*$ (uniqueness of the solution of $LCP(M,q)$) then

$$|x_1^*| + x_1^* = |x_2^*| + x_2^*. \tag{4}$$

In the same way, for w_1^* and w_2^*, we obtain

$$|x_1^*| - x_1^* = |x_2^*| - x_2^* \tag{5}$$

so (4)–(5) means that $x_1^* = x_2^*$.

Now, we give the following algorithm for solving $\tilde{F}(p,x) = 0$ (see [26]):

Algorithm. *Step* 0: Determine $\epsilon, p, k^*, \rho, \sigma$ such that k^* is a positive integer, $0 < \rho < 1/2$, and $\rho < \sigma < 1$;
Step 1: Select two points $x^{(0)}$ and $x^{(1)} \in \mathrm{IR}^n$;
Step 2: For $k = 1, 2, \ldots$ until termination, do the following:

1. Compute the steepest descent direction
   $$d^{(k)} := -J(p, x^{(k)})^T \tilde{F}(p, x^{(k)}),$$
   where
   $$\tilde{F}(p,x) := (M+I)x + \frac{1}{p}(M-I)\ln(e^0 + e^{px} + e^{-px}) + q;$$
   $$J(p,x) := (M+I) + (M-I)E(p,x)$$
   and
   $$E_{ij}(p,x) := \delta_{ij}\frac{e^{px_i} - e^{-px_i}}{1 + e^{px_i} + e^{-px_i}};$$
   we recall that $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.
2. If k equals a multiple of k^*, then insert a steepest descent direction step, that is, let $s^{(k)} := d^{(k)}$ and go to step 2.7;
3. Compute:
   $$\begin{cases} u^{(k)} := \xi_1 \tilde{F}(p, x^{(k)}) \\ v^{(k)} := \xi_2 (x^{(k)} - x^{(k-1)}) \end{cases}$$
   with
   $$\xi_1 = -\frac{\|x^{(k)} - x^{(k-1)}\|^2}{\langle x^{(k)} - x^{(k-1)}, \tilde{F}(p, x^{(k)}) - \tilde{F}(p, x^{(k-1)})\rangle}$$
   and
   $$\xi_2 = -\frac{\langle \tilde{F}(p, x^{(k)}) - \tilde{F}(p, x^{(k-1)}), \tilde{F}(p, x^{(k)})\rangle}{\|\tilde{F}(p, x^{(k)}) - \tilde{F}(p, x^{(k-1)})\|^2}.$$
4. If $(u^{(k)} - v^{(k)})^T d^{(k)} \neq 0$, then choose
   $$\alpha^{(k)} > \frac{-\langle v^{(k)}, d^{(k)}\rangle}{\langle u^{(k)} - v^{(k)}, d^{(k)}\rangle}$$
   such that $\alpha^{(k)}$ maximizes the value of
   $$\mathrm{Cos}[s^{(k)}, d^{(k)}] = \frac{\langle s^{(k)}, d^{(k)}\rangle}{\|s^{(k)}\|.\|d^{(k)}\|};$$
   set
   $$s^{(k)} := \alpha^{(k)} u^{(k)} + (1 - \alpha^{(k)}) v^{(k)}$$
   and go to step 2.7;

Table 1
Approximations of the solution of Example 1 by two methods.

	Iteration	z_1	z_2	z_3	z_4
Fixed point method	$k = 01$	0.0000000	0.0000000	0.0000000	0.0000000
	$k = 05$	0.7883251	0.0000000	1.3448593	0.0000000
	$k = 10$	1.0197946	0.0000000	0.9884737	0.0000000
	$k = 15$	0.9985643	0.0000000	1.0012907	0.0000000
	$k = 20$	1.0001030	0.0000000	0.9999377	0.0000000
	$k = 25$	0.9999918	0.0000000	1.0000058	0.0000000
	$k = 30$	1.0000005	0.0000000	0.9999997	0.0000000
	$k = 33$	0.9999999	0.0000000	1.0000001	0.0000000
	$k = 34$	1.0000001	0.0000000	1.0000000	0.0000000
	$k = 35$	1.0000000	0.0000000	1.0000000	0.0000000
Using vector divisions method	$k = 01$	0.0000000	0.0000000	0.0000000	0.0000000
	$k = 02$	4.0000000	0.0000000	4.0000000	0.0000000
	$k = 03$	1.0000000	0.0000000	1.0000000	0.0000000

5. If $(u^{(k)} - v^{(k)})^T d^{(k)} = 0$ and $\langle v^{(k)}, d^{(k)}\rangle > 0$,
   then set $s^{(k)} := \frac{u^{(k)}+v^{(k)}}{2}$ and go to step 2.7;
6. If $(u^{(k)} - v^{(k)})^T d^{(k)} = 0$ and $\langle v^{(k)}, d^{(k)}\rangle \leqslant 0$,
   then set $s^{(k)} := d^{(k)}$ and go to step 2.7;
7. Take a line search along the direction $s^{(k)}$ to determine the step length γ such that
   $$f(p, x^{(k)} + \gamma s^{(k)}) \leqslant f(p, x^{(k)}) - \gamma\rho\langle d^{(k)}, s^{(k)}\rangle$$
   and
   $$\langle \nabla f(p, x^{(k)} + \gamma s^{(k)}), s^{(k)}\rangle \geqslant -\sigma\langle d^{(k)}, s^{(k)}\rangle;$$
   where
   $$f(p, x) = \frac{1}{2}\|\tilde{F}(p, x)\|^2$$
8. Set $x^{(k+1)} := x^{(k)} + \gamma s^{(k)}$ and go to the next iteration.

5. Numerical examples

In this section, we consider two examples to test the speed of convergence of our algorithm and its efficiency to solve a large scale problem. In a first example, we exhibit the number of iterations needed to obtain an approximation with 6 digits of the known solution and we compare it with the method based on the "Fixed Point". In the second example, the matrix M_n of the problem is variable with the dimension n. For eight values of n between 10 and 2000 we give the number of iterations and the CPU time needed to obtain an approximation of the solution of each $LCP(M_n, q)$.

Example 1. Consider the following linear complementarity problem: Find a vector z in IR^4 satisfying $z^T(Mz + q) = 0$, $Mz + q \geqslant 0$ and $z \geqslant 0$, where

$$M = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} -4 \\ 3 \\ -4 \\ 2 \end{bmatrix}.$$

It is easy to see that the above matrix M is positive definite.

We apply the "Fixed Point" and "Using Vector Divisions" methods to solve this example and compare it with the known exact solution $z^* = (1, 0, 1, 0)^T$.

When looking for an approximation with six significant digits, we obtain that (see Table 1), using the vector divisions method requires only 3 iterations and $CPU\ time = 0.0010014$ s. The Fixed Point method requires at least 35 iterations and $CPU\ time = 0.653$ s to achieve the same result.

Example 2. Consider the following class of linear complementarity problems [27]: For a given integer n, find a vector z in IR^n satisfying $z^T(Mz + q) = 0$, $Mz + q \geqslant 0$ and $z \geqslant 0$, where

$$M = \begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 2 & 5 & 6 & \dots & 6 \\ 2 & 6 & 9 & \dots & 10 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 6 & 10 & \dots & 4(n-1)+1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} -1 \\ -1 \\ \dots \\ -1 \end{bmatrix}.$$

It is possible to prove that the above matrix M is positive definite.

Table 2
Number of iterations and CPU time with our method for Example 2 where $p = 30, x^{(0)} = (-1, 2, -3, \dots, n(-1)^n)^T$.

n	CPU	k	$\|\tilde{F}(p, x)\|$	$z^T(Mz+q)$
10	0.473	6	$6.77\ 10^{-7}$	$2.98\ 10^{-4}$
50	0.753	48	$8.43\ 10^{-7}$	$4.72\ 10^{-4}$
100	1.033	76	$8.85\ 10^{-7}$	$6.78\ 10^{-4}$
200	1.313	104	$4.13\ 10^{-7}$	$5.18\ 10^{-4}$
500	1.593	133	$7.12\ 10^{-7}$	$3.03\ 10^{-4}$
1000	1.873	161	$5.58\ 10^{-7}$	$6.42\ 10^{-4}$
1500	2.153	189	$2.76\ 10^{-7}$	$3.89\ 10^{-4}$
2000	2.433	218	$6.13\ 10^{-9}$	$1.30\ 10^{-4}$

As mentioned previously, for p large enough, the unique solution x^* of $\tilde{F}(p, x^*) = 0$ is an approximation of the unique solution z^* ($z^* = |x^*| + x^*$) of the linear complementarity problem. For this example we take $p = 30, x^{(0)} = (-1, 2, -3, \dots, n(-1)^n)^T$ and we test our method with different values of n.

The method was coded in Matlab 7.11 and run on a personal computer with a 2.5 GHz CPU processor. We stop the iterations if the condition $\|\tilde{F}(p, x^{(k)})\| \leqslant 10^{-6}$ is satisfied. Table 2 lists the results of this simulation.

In this table, "n" denotes the dimension of the matrix M; "*CPU*" denotes the computer time used (in second) and k means the total number iterations needed to achieve $\|\tilde{F}(p, x)\| < 10^{-6}$.

We note that starting from each initial point, the method terminates at the solution of the problem successfully. The proposed method can solve large-scale linear complementarity problems efficiently.

6. Conclusion

In this paper we have used the fact that solving the linear complementarity problem is completely equivalent to solving the nonlinear equation $F(x) = 0$ where F is a function from IR^n into itself defined by $F(x) = (M + I)x + (M - I)|x| + q$. For solving this equation we develop an algorithm based on the Shi's method. We use vector divisions with the secant method, to obtain a globally convergent hybrid algorithm for solving this equation. We start by building a sequence of smooth functions $\tilde{F}(p, x)$ which is uniformly convergent to the function $F(x)$; and we show that the zeros of the sequence of the functions $\tilde{F}$ give a good approximation of the zero of the function F for a fixed value of a parameter p large enough. Some computational experiments are presented to indicate that the method gives good results and that the algorithm can be used for solving large-scale linear complementarity problems efficiently.

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